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Isomorphism Transformation Between the Hydrogen Atom and Four-Dimensional Harmonic Oscillator

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By means of the $SU(1, 1)$ algebra we study the relationships between the coordinates, energy, angular momentum, spherical harmonics, and radial function of a hydrogen atom and those of a four-dimensional harmonic oscillator. The energies and angular momenta of the two quantum systems are found to correspond to one another.

1. INTRODUCTION

The relationship between a hydrogen atom and a four-dimensional (4D) harmonic oscillator has been studied by many authors (Ravdal and Toyoda, 1967; Ikeda and Miyachi, 1970; Bioteux, 1972, 1973a, b; Barut *et al.*, Chen, 1980a, b, 1981a, b; Iwai, 1982; Kibler and Negadi, 1983a, b, 1984), but there are still problems in need of further study. For example, what is the relationship between the angular momenta of the two quantum systems? The so-called KS transformation (Kibler and Negadi, 1983a, b, 1984)

$$z_1 = 2(v_1v_4 + v_2v_3), \quad z_2 = 2(v_2v_4 - v_1v_3), \quad z_3 = v_1^2 + v_2^2 - v_3^2 - v_4^2 \tag{1}$$

where z_i ($i = 1, 2, 3$) and v_α ($\alpha = 1, 2, 3, 4$) are the coordinates of the

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hydrogen atom and the 4D harmonic oscillator, respectively, has been used to transform the energy eigenvalue equation of the hydrogen atom

$$\left(-\frac{\hbar^2}{2\mu}\Delta_z - \frac{Ze^2}{z}\right)\psi = E\psi \tag{2}$$

into that of the 4D harmonic oscillator

$$\left(-\frac{\hbar^2}{2\mu}\Delta_v - 4Ev^2\right)\psi = 4Ze^2\psi \tag{3}$$

accompanied by the constraint

$$\left(v_1\frac{\partial}{\partial v_2} - v_2\frac{\partial}{v_1} - v_3\frac{\partial}{\partial v_4} + v_4\frac{\partial}{\partial v_3}\right)\psi = 0 \tag{4}$$

where $z = (z_i\ z_i)^{1/2}$, $v^2 = v_\alpha v_\alpha$, $\Delta_z = \partial^2/\partial z_i \partial z_i$, and $\Delta_v = \partial^2/\partial v_\alpha \partial v_\alpha$. In this transformation, the same function ψ is used to represent the energy eigenfunctions of both quantum systems, and the energies of the two quantum systems do not correspond to one another. It is not clear whether the different energy eigenstates of the hydrogen atom can be transformed into the different energy eigenstates of the 4D harmonic oscillator.

In this paper, we, too, study the relationship between the hydrogen atom and the 4D harmonic oscillator, but with a method that is different from the one used by other authors. Zeng *et al.* (1994a, b) used the $SU(1, 1)$ algebra to give a simple and general algebraic relationship between a d-dimensional hydrogen atom and a D-dimensional harmonic oscillator in which the energies of the two quantum systems correspond. In Zeng *et al.* (1997) those authors further developed their method and proved strictly that the coordinate transformation between the d-dimensional hydrogen atom and the D-dimensional harmonic oscillator forms a Clifford algebra of traceless. Here use the method of Zeng *et al.* (1994a, b) to study the relation between the hydrogen atom and the 4D harmonic oscillator. The method is very simple and effective, can help solve the problems about the relationship between the hydrogen atom and the 4D harmonic oscillator.

We first point out that in studying the relationship between the hydrogen atom and the 4D harmonic oscillator, it is convenient to use dimensionless variables, which are chosen to be $x_i = z_i/a_0$ ($i = 1, 2, 3$) for the hydrogen atom and $v_\alpha = v_\alpha/v_0$ ($\alpha = 1, 2, 3, 4$) for the 4D harmonic oscillator, where z_i and v_α are the coordinates in the usual spaces of the hydrogen atom and the 4D harmonic oscillator, respectively, $a_0 = \hbar^2/(\mu e^2)$ is the first Bohr orbit radius of the hydrogen atom, and $v_0 = \sqrt{\hbar/\mu\omega}$.

Our method is as follows.

First, we consider the space $\{x_j\ j = 1, 2, 3,\}$, which is the coordinate space of the hydrogen atom. In it, we construct the set of operators

$$K_1 = \frac{1}{2}(x\Delta_x + x), \quad K_2 = i\left(1 + x_j\frac{\partial}{\partial x_j}\right), \quad K_3 = -\frac{1}{2}(x\Delta_x - x) \tag{5}$$

where $x = (x_j x_j)^{1/2}$ and $\Delta_x = \partial^2/\partial x_j \partial x_j$. These operators constitute the $SU(1, 1)$ algebra, because they satisfy the commutation relations

$$[K_1, K_2] = -iK_3, \qquad [K_2, K_3] = iK_1, \qquad [K_3, K_1] = iK_2 \tag{6}$$

Next, we consider the space $\{u_\alpha, \alpha = 1, 2, 3, 4\}$, which is the coordinate space of the 4D harmonic oscillator. In it, we construct the set of operators also denoted by K_1, K_2, and K_3

$$K_1 = \frac{1}{4}(u^2 + \Delta_u), \quad K_2 = i\left(1 + \frac{1}{2}u_\alpha\frac{\partial}{\partial u_\alpha}\right), \quad K_3 = \frac{1}{4}(u^2 - \Delta_u) \tag{7}$$

where $u = (u_\alpha u_\alpha)^{1/2}$ and $\Delta_u = \partial^2/\partial u_\alpha \partial u_\alpha$. It is easy to show that these operators also satisfy the communication relations (6); then they also constitute the $SU(1, 1)$ algebra.

The two sets of the operators (5) and (7) satisfy the $SU(1, 1)$ algebraic relations. Thus, the $SU(1, 1)$ algebra builds a bridge between the hydrogen atom and the 4D harmonic oscillator by which all the problems (coordinates, energy, angular momentum, spherical harmonics, radial function, etc.) of the hydrogen atom may be transformed into the corresponding problems of the 4D harmonic oscillator.

The contents of this paper are as follows: In Section 2 we determine the relationship between the coordinates of the hydrogen atom and the 4D harmonic oscillator. This relationship is similar to the so-called *KS* transformation, but is not exactly the same. In Section 3 the coordinate transformation obtained in Section 2 is used to study the relationship between the angular momenta of the hydrogen atom and the 4D harmonic oscillator, and the relationship between a three-dimensional spherical harmonics and a four-dimensional spherical harmonics. In Section 4 we study the relationships between the energy and the energy eigenstate of the hydrogen atom and those of the 4D harmonic oscillator. The result we obtain is that the different energies and energy eigenstates of the hydrogen atom correspond to the different energies and energy eigenstates of the 4D harmonic oscillator. Finally, in Section 5 we use the results in Sections 3 and 4 to determine the relationship between the radical functions of the hydrogen atom and the 4D harmonic oscillator. The result shows that the two radical functions in fact can be connected by a simple transformation.

2. TRANSFORMATION BETWEEN COORDINATES

Now we determine the transformation between the coordinates of the hydrogen atom and the 4D harmonic oscillator. We have shown that both operators (5) (which are defined by the coordinates of the hydrogen atom) and operators (7) (which are defined by the coordinates of the 4D harmonic oscillator) constitute the *SU*(1,1) algebra. Thus, identifying the operators (5) with the operators (7), we have the following operator equations:

$$x\Delta_x + x = \frac{1}{2}(\Delta_u + u^2) \tag{8}$$

$$x_j \frac{\partial}{\partial x_j} = \frac{1}{2} u_\alpha \frac{\partial}{\partial u_\alpha} \tag{9}$$

$$x\Delta_x - x = \frac{1}{2}(\Delta_u - u^2) \tag{10}$$

It is clear that (8) and (10) may be rewritten as

$$x = \frac{1}{2} u^2 \tag{11}$$

$$x\Delta_x = \frac{1}{2}\Delta_u \tag{12}$$

Using the operator equations (8)–(12), one can determine the transformation between the coordinates of the hydrogen atom and the 4D harmonic oscillator. In fact, one can know that x_j are certainly homogeneous functions of u_α with degree two in accordance with equation (11), while the specific forms of the functions may be determined with the help of the other operator equations. Of course, the forms of the functions are not unique.

We give the following form for the coordinate transformation between the hydrogen atom and the 4D harmonic oscillator:

$$x_1 = u_1u_4 + u_2u_3, \; x_2 = u_2u_4 - u_1u_3, \; x_3 = \frac{1}{2}(u_1^2 + u_2^2 - u_3^2 - u_4^2) \tag{13}$$

It is not difficult to check the correctness of the transformation (13). The method is to substitute (13) into (11) or act with the operator equations (8)–(12) on (13).

The transformation (13) differs in form from the so-called *KS* transformation by only the constant 2, but this difference is very interesting. From the transformation (13), one can determine all the relationships between the hydrogen atom and the 4D harmonic oscillator.

It is important to note that, given the transformation (13), one can derive the operator equations in the opposite direction. In fact, the operator equation (9) can be derived from the transformation (13), but the operator equation (12) in this case should be replaced by

$$x\Delta_x = \frac{1}{2}\Delta_u - \frac{1}{2}u^{-2}P^2 \tag{14}$$

where

$$P = u_1\frac{\partial}{\partial u_2} - u_2\frac{\partial}{\partial u_1} - u_3\frac{\partial}{\partial u_4} + u_4\frac{\partial}{\partial u_3} \tag{15}$$

It seems that equation (14) is not similar to equation (12) in form. But noting

$$Px_j = 0 \qquad (j = 1, 2, 3) \tag{16}$$

one easily sees that equation (14) is consistent with equation (12). Because P is a differential operator, (16) may be generalized to an arbitrary function $\Psi(\mathbf{x})$ of x_i, i.e., one has

$$P\Psi(\mathbf{x}) = \mathbf{0} \tag{17}$$

provided the variables x_i in $\Psi(\mathbf{x})$ can be expressed in terms of u_α in the form (13), or $\Psi(\mathbf{x})$ can be expressed as a function $\Psi'(\mathbf{u})$ of u_α by the help of the transformation (13), so one has $\Psi'(\mathbf{u}) = \Psi(\mathbf{x})$. In considering the action of the operator P on this function relation, one may set $P = 0$.

We shall study the relationship between the angular momenta of the hydrogen atom and the 4D harmonic oscillator. To this end, now we use spherical coordinates to describe the transformation between the coordinates of the two quantum systems.

The spherical coordinates (x, θ, ϕ) of the hydrogen atom are defined by

$$x_1 = x\sin\theta\sin\phi, \qquad x_2 = x\sin\theta\cos\phi, \qquad x_3 = \cos\theta \tag{18}$$

where $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. The spherical coordinates $(u, \theta_2, \theta_1, \theta_0)$ of the 4D harmonic oscillator are defined by

$$\begin{aligned} u_1 &= u\sin\theta_2\sin\theta_1\sin\theta_0, \qquad & u_2 &= u\sin\theta_2\sin\theta_1\cos\theta_0 \\ u_3 &= u\sin\theta_2\cos\theta_1 & u_4 &= u\cos\theta_2 \end{aligned} \tag{19}$$

where $0 \leq \theta_2 \leq \pi$, $0 \leq \theta_1 \leq \pi$, and $0 \leq \theta_0 \leq 2\pi$. Substituting equations (18) and (19) into equations (13), we find again that the equation (11) holds, and obtain the relations between the angular coordinates of the two quantum systems,

$$x = \frac{1}{2}u^2, \qquad \cos\frac{\theta}{2} = \sin\theta_2 \sin\theta_1, \qquad \phi = \theta_0 + \psi \tag{20}$$

where $\tan\psi = \tan\theta_2 \cos\theta_1$. The transformation (20) is very simple and useful; we shall use it to obtain the relation between the three-dimensional and four-dimensional spherical harmonics.

3. TRANSFORMATION BETWEEN ANGULAR MOMENTA

In order to study the connection between the angular momenta of the hydrogen atom and the 4D harmonic oscillator, we first describe separately the angular momenta of the two quantum systems.

The angular momentum operators of the hydrogen atom are defined by

$$L_{jk} = -i\left(x_j \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_j}\right) \tag{21}$$

Its square is $L^2 = \Sigma_{i<j} L_{ij} L_{ij}$. If one sets $L_1 = L_{23}$, $L_2 = L_{31}$, and $L_3 = L_{12}$, then one has

$$[L_1, L_2] = iL_3, \qquad [L_2, L_3] = iL_1, \qquad [L_3, L_1] = iL_2 \tag{22}$$

which means that the L_i construct the $SO(3)$ algebra.

One usually chooses L^2 and L_3 as a complete set of operators. Their common eigenfunctions are the spherical harmonics $Y_{lm}(\theta, \phi)$ in the three-dimensional space,

$$L^2 Y_{lm}(\theta, \phi) = l(l+1)Y_{lm}(\theta, \phi), \qquad L_3 Y_{lm}(\theta, \phi) = mY_{lm}(\theta, \phi) \tag{23}$$

where $l = 0, 1, 2, \ldots$; $m, = l, l-1, \ldots, -l$. The spherical coordinates $[x = (x_i x_i)^{1/2}, \theta, \phi]$ we use here are defined by (18), and are not the same as those used in quantum mechanics (Messiah, 1972).[3]

The angular momentum operators of the 4D harmonic oscillator are defined by

$$L'_{\alpha\beta} = -i\left(u_\alpha \frac{\partial}{\partial u_\beta} - u_\beta \frac{\partial}{\partial u_\alpha}\right) \tag{24}$$

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[3] Note that the angle ϕ in $Y_{lm}(\theta, \phi)$ is different from the angle φ in $Y_{lm}(\theta, \varphi)$ used in quantum mechanics; one has the relation $\varphi = \pi/2 - \phi$ between them.

Its square is $L'^2 = \Sigma_{\alpha<\beta} L'_{\alpha\beta} L'_{\alpha\beta}$. The $L'_{\alpha\beta}$ satisfy

$$[L'_{\gamma\sigma}, L'_{\alpha\beta}] = i\delta_{\alpha\gamma} L'_{\beta\sigma} - i\delta_{\alpha\sigma} L'_{\beta\gamma} - i\delta_{\beta\gamma} L'_{\alpha\sigma} + i\delta_{\beta\sigma} L'_{\alpha\gamma} \tag{25}$$

which means that the $L'_{\alpha\beta}$ construct the $SO(4)$ algebra. In (25), if one sets $\alpha = \sigma$, then one gets

$$[L'_{\alpha\beta}, L'_{\gamma\alpha}] = iL'_{\beta\gamma} \tag{26}$$

One may set $A_1 = L'_{23}$, $A_2 = L'_{31}$, and $A_3 = L'_{12}$. Clearly the A_i construct the $SO(3)$ algebra, a subalgebra in the $SO(4)$ algebra constructed by $L'_{\alpha\beta}$, since they satisfy

$$[A_1, A_2] = iA_3, \qquad [A_2, A_3] = iA_1, \qquad [A_3, A_1] = iA_2 \tag{27}$$

For the $SO(4)$ algebra constructed by $L'_{\alpha\beta}$, one usually chooses L'^2, A^2, and A_3 as a complete set of operators; their common eigenfunctions, the spherical harmonics in the 4D space, are well known (Qian and Zeng, 1993). But we do not make this choice in this paper.

We set

$$B_1 = \frac{1}{2}(L'_{23} + L'_{14}), \qquad B_2 = \frac{1}{2}(L'_{31} + L'_{24}), \qquad B_3 = \frac{1}{2}(L'_{12} + L'_{34}) \tag{28}$$

It is easy to show that B_i also construct the $SO(3)$ algebra, i.e., satisfy

$$[B_1, B_2] = iB_3, \qquad [B_2, B_3] = iB_1, \qquad [B_3, B_1] = iB_2 \tag{29}$$

Its square is

$$B^2 = \frac{1}{4} L'^2 + \frac{1}{2} P_1 \tag{30}$$

where $P_1 = L'_{12}L'_{34} + L'_{23}L'_{14} + L'_{31}L'_{24}$.

For the $SO(4)$ algebra constructed by $L'_{\alpha\beta}$, we choose $L'2$, B^2, and B_3 as a complete set of operators. Their common eigenfunctions, the spherical harmonics in the 4D space, are denoted by $Y_{l'_2, l'_1, m'}(\theta_2, \theta_1, \theta_0)$ that is, we have

$$\begin{aligned} L'2Y_{l'_2, l'_1, m'}(\theta_2, \theta_1, \theta_0) &= l'_2(l'_2 + 2)Y_{l'_2, l'_1, m'}(\theta_2, \theta_1, \theta_0) \\ B^2Y_{l'_2, l'_1, m'}(\theta_2, \theta_1, \theta_0) &= l'_1(l'_1 + 1)Y_{l'_2, l'_1, m'}(\theta_2, \theta_1, \theta_0) \\ B_3Y_{l'_2, l'_1, m'}(\theta_2, \theta_1, \theta_0) &= m'Y_{l'_2, l'_1, m'}(\theta_2, \theta_1, \theta_0) \end{aligned} \tag{31}$$

where $l'_2, l'_1 = 0, 1, 2, \ldots\ 0 \le l'_1 \le l'_2$, $m' = l'_1, l'_1 - 1, \ldots, -l'_1$. The $Y_{l'_2, l'_1, m'}(\theta_2, \theta_1, \theta_0)$ are different from those defined in Qian and Zeng (1993), but may be obtained through the transformation of the latter. The spherical coordinates $[u = (u_\alpha u_\alpha)^{1/2}, \theta_2, \theta_1, \theta_0]$ used here are defined by (19).

Now we determine the relations between the angular momentum operators of the hydrogen atom and the 4D harmonic oscillator.

From the transformation (13), we obtain the differential operator relations

$$\begin{aligned}
\frac{\partial}{\partial x_1} &= u^{-2}\left(u_4 \frac{\partial}{\partial u_1} + u_3 \frac{\partial}{\partial u_2} + u_2 \frac{\partial}{\partial u_3} + u_1 \frac{\partial}{\partial u_4}\right) \\
\frac{\partial}{\partial x_2} &= u^{-2}\left(-u_3 \frac{\partial}{\partial u_1} + u_4 \frac{\partial}{\partial u_2} - u_1 \frac{\partial}{\partial u_3} + u_2 \frac{\partial}{\partial u_4}\right) \\
\frac{\partial}{\partial x_3} &= u^{-2}\left(u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2} - u_3 \frac{\partial}{\partial u_3} - u_4 \frac{\partial}{\partial u_4}\right)
\end{aligned} \tag{32}$$

Using the relations (13) and (32), we can express L_{jk} in terms of $L'_{\alpha\beta}$ as follows:

$$L_{12} = -L_{21} = u^{-2}\{(u_1^2 + u_2^2)\, L'_{34} + (u_3^2 + u_4^2)L'_{12}\} \tag{33}$$

$$\begin{aligned}
L_{13} &= -L_{31} \\
&= -\frac{1}{2} u^{-2}\{(u_1^2 + u_4^2)L'_{14} + (u_2^2 + u_3^2)L'_{23} \\
&\quad + (u_2u_4 - u_1u_3)(-L'_{12} + L'_{34}) + (u_1u_2 + u_3u_4)(L'_{24} + L'_{13})\}
\end{aligned} \tag{34}$$

$$\begin{aligned}
L_{23} = -L_{32} &= -\frac{1}{2} u^{-2}\{(u_2^2 + u_4^2)L'_{24} - (u_1^2 + u_3^2)L'_{13} \\
&\quad - (u_1u_2 - u_3u_4)(L'_{23} - L'_{14}) - (u_2u_3 + u_1u_4)(-L'_{12} + L'_{34})\}
\end{aligned} \tag{35}$$

Introducing the relations (33)–(35) into L^2 and making some complicated operations, we obtain the connection between the operators L^2 and $L'2$ as

$$L^2 = \frac{1}{4}(L'2 - P_2^2) \tag{36}$$

where $P_2 = -L'_{12} + L'_{34}$. Relations (33)–(36) are just the transformations between the angular momentum operators of the hydrogen atom and the 4D harmonic oscillator.

One feature of the transformation (13) which is very important in determining the connection between the angular momenta of the hydrogen atom and the 4D harmonic oscillator is that the actions of the operator P_2 on x_i give the result 0:

$$P_2 x_i = 0 \tag{37}$$

P_2 is a differential operator; therefore, the result (37) may be generalized to an arbitrary function $Y(\mathbf{x})$, provided the variables x_i in $Y(\mathbf{x})$ can be expressed in terms of u_α in the form (13), i.e., one has $P_2 Y(\mathbf{x}) = 0$.

We now assume that there are two functions $Y(\mathbf{x})$ and $Y'(\mathbf{u})$, which may be considered as the state functions of the hydrogen atom and the 4D harmonic oscillator, respectively. We assume also that $Y'(\mathbf{u})$ comes from the transformation of $Y(\mathbf{x})$ via the coordinate transformation (13), and thus one has $Y'(\mathbf{u}) = Y(\mathbf{x})$. It is clear that the action of the operator P_2 on $Y'(\mathbf{u})$ also gives the result 0: $P_2 Y'(\mathbf{u}) = 0$. The physical meaning of this result is that the state function of the 4D harmonic oscillator connected with the hydrogen atom is such a function, in which the components L'_{12} and L'_{34} of the angular momentum operator of the 4D harmonic oscillator are the same.

Our operator equations always act on the function relation $Y'(\mathbf{u}) = Y(\mathbf{x})$. The discussion above shows that in the operator equations (33)–(36), we can always set $P_2 = 0$ or $L'_{12} = L'_{34}$. In other words, the operator equations (33)–(36) can be written simply as

$$L_{12} = L'_{12} \tag{38}$$

$$L_{13} = -\frac{1}{2} u^{-2}\{(u_1^2 + u_4^2)L'_{14} + (u_2^2 + u_3^2)L'_{23} + (u_1u_2 + u_3u_4)(L'_{24} + L'_{13})\} \tag{39}$$

$$L_{23} = -\frac{1}{2} u^{-2}\{(u_2^2 + u_4^2)L'_{24} - (u_1^2 + u_3^2)L'_{13} - (u_1u_2 - u_3u_4)(L'_{23} - L'_{14})\} \tag{40}$$

$$L^2 = \frac{1}{4} L'^2 \tag{41}$$

Now we consider the operator B^2 defined in (30). It is easy to show that the action of the operator P_1 on x_i also gives the result 0: $P_1 x_i = 0$. Thus, in considering the action of B^2 on $Y'(\mathbf{u}) = Y(\mathbf{x})$, one may set also $P_1 = 0$, so one has $B^2 = \frac{1}{4}L'^2$. Comparing this relation with (41), one finds

$$L^2 = B^2 \tag{42}$$

According to (41) and (42) and noting that $B_3 = L'_{12} = L'_{34}$, if $Y(\mathbf{x})$ is the common eigenfunction of L^2 and L_3 with the eigenvalues $l(l+1)$ and m, respectively, i.e., it satisfies

$$L^2 Y(\mathbf{x}) = l(l+1)Y(\mathbf{x}), \qquad L_3 Y(\mathbf{x}) = mY(\mathbf{x}) \tag{43}$$

then $Y'(\mathbf{u})$ is certainly the common eigenfunction of L'^2, B^2, and B_3, with the eigenvalues $l'_2(l'_2+2)$, $l'_1(l'_1+1)$, and m', respectively, i.e., it satisfies

$$\begin{aligned} L'^2Y'(\mathbf{u}) &= l'_2(l'_2+2)Y'(\mathbf{u}) \\ B^2Y'(\mathbf{u}) &= l'_1(l'_1+1)Y'(\mathbf{u}) \\ B_3Y'(\mathbf{u}) &= m'Y'(\mathbf{u}) \end{aligned} \tag{44}$$

Moreover, one easily shows that

$$l'_2 = 2l'_1, \qquad l'_1 = l, \qquad m' = m \tag{45}$$

which represents the relationship between the angular momenta of the hydrogen atom and the 4D harmonic oscillator.

Let us express the results above in spherical coordinates. In this case, the coordinate transformation becomes (20). The transformation relationship between the angular momentum operators of the hydrogen atom and the 4D harmonic oscillator may be rewritten in spherical coordinates, but the relations $L'2 = 4L^2$, $B^2 = L^2$, and B_3 ($= L'_{12}$ or L'_{34}) $= L_3$ naturally are unchanged.

In spherical coordinates, the common eigenfunction of L^2 and L_3 is the three-dimensional spherical harmonic $Y_{lm}(\theta, \phi)$, which is a function of the angles θ and ϕ and is labeled by two quantum numbers l and m, in other words, $Y_{lm}(\theta, \phi)$ satisfies (23). If we transform the angles θ and ϕ in $Y_{lm}(\theta, \phi)$ into the angles θ_2, θ_1, and θ_0 in accordance with (20), then Y_{lm} (θ, ϕ) changes to the 4D spherical harmonics $Y'_{2l,l,m}(\theta_2, \theta_1, \theta_0)$, which is a function of the angles θ_2, θ_1, and θ_0; in labeling it, we here have used the quantum numbers $2l$, l, and m. Then one has

$$Y'_{2l,l,m}(\theta_2, \theta_1, \theta_0) = Y_{lm}(\theta, \phi) \tag{46}$$

According to the description above, $Y'_{2l,l,m}(\theta_2, \theta_1, \theta_0)$ is certainly the common eigenfunction of $L'2$, B^2, and B_3; the corresponding eigenvalues are $2l(2l + 2)$, $l(l + 1)$, and m, respectively; in other words, it satisfies

$$\begin{aligned} L'^2Y'_{2l,l,m}(\theta_2, \theta_1, \theta_0) &= 2l(2l+2)Y'_{2l,l,m}(\theta_2, \theta_1, \theta_0) \\ B^2Y'_{2l,l,m}(\theta_2, \theta_1, \theta_0) &= l(l+1)Y'_{2l,l,m}(\theta_2, \theta_1, \theta_0) \\ B_3Y'_{2l,l,m}(\theta_2, \theta_1, \theta_0) &= mY'_{2l,l,m}(\theta_2, \theta_1, \theta_0) \end{aligned} \tag{47}$$

However, because $Y'_{2l,l,m}(\theta_2, \theta_1, \theta_0)$ and $Y_{lm}(\theta, \phi)$ are spherical harmonics in two different spaces, $Y'_{2l,l,m}$ $(\theta_2, \theta_1, \theta_0)$ is not normalized if $Y_{lm}(\theta, \phi)$ is already normalized.

The spherical function $Y_{l'_2,l'_1,m'}(\theta_2, \theta_1, \theta_0)$ has been defined as the common eigenfunction of $L'2$, B^2, and B_3 [see (31)]. It clearly may be used to express the angular momentum state of the 4D harmonic oscillator connected to the hydrogen atom, provided we set $l'_2 = 2l$, $l'_1 = l$, and $m' = m$, and can be obtained by setting

$$Y_{2l,l,m}(\theta_2, \theta_1, \theta_0) = N_1 Y'_{2l,l,m}(\theta_2, \theta_1, \theta_0) \tag{48}$$

where N_1 is a normalization constant. Thus, we finally have

$$Y_{2l,l,m}(\theta_2, \theta_1, \theta_0) = N_1 Y_{lm}(\theta, \phi) \tag{49}$$

This is just the relationship between the spherical harmonics of the 4D harmonic oscillator and the hydrogen atom.

$Y_{2l,l,m}(\theta_2, \theta_1, \theta_0)$ includes only two quantum numbers l and m. In other words, the spherical harmonics of the 4D harmonic oscillator connected by the hydrogen atom may be labeled by using only two quantum numbers. The reason for this situation is that our 4D harmonic oscillator comes from the transformation of the hydrogen atom; thus the components L'_{34} and L'_{12} of its angular momentum operator are the same.

We give an example. The spherical harmonics of the hydrogen atom in the case $l = 1$ are (Messiah, 1972)

$$Y_{11}(\theta, \phi) = \sqrt{\frac{3}{8\pi}} \sin\theta e^{i(\pi/2-\phi)}$$

$$Y_{10}(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$Y_{1-1}(\theta, \phi) = \sqrt{\frac{3}{8\pi}} \sin\theta\, e^{-i(\pi/2-\phi)}$$

which are normalized. Using the angle transformation (20) and multiplying by $N_1 = \sqrt{2/\pi}$, they are changed into the spherical harmonics of the 4D harmonic oscillator

$$Y_{211}(\theta_2, \theta_1, \theta_0) = \frac{\sqrt{3}}{\pi}(1 - \sin^2\theta_2 \sin^2\theta_1)^{1/2} \sin\theta_2 \sin\theta_1\, e^{i(\pi/2-\theta_0-\psi)}$$

$$Y_{210}(\theta_2, \theta_1, \theta_0) = \sqrt{\frac{3}{2}}\,\frac{1}{\pi}(2\sin^2\theta_2 \sin^2\theta_1 - 1) \tag{51}$$

$$Y_{21-1}(\theta_2, \theta_1, \theta_0) = \frac{\sqrt{3}}{\pi}(1 - \sin^2\theta_2 \sin^2\theta_1)^{1/2} \sin\theta_2 \sin\theta_1\, e^{-i(\pi/2-\theta_0-\psi)}$$

It is easy to check that $Y_{21m}(\theta_2, \theta_1, \theta_0)$ satisfy the relations

$$\begin{aligned} L'^2 Y_{21m}(\theta_2, \theta_1, \theta_0) &= 8Y_{21m}(\theta_2, \theta_1, \theta_0) \\ B^2 Y_{21m}(\theta_2, \theta_1, \theta_0) &= 2Y_{21m}(\theta_2, \theta_1, \theta_0) \\ B_3 Y_{21m}(\theta_2, \theta_1, \theta_0) &= mY_{21m}(\theta_2, \theta_1, \theta_0) \end{aligned} \tag{52}$$

and are normalized, which shows the correctness of the result (49). In order to check (52), one needs to use the following expressions for L'^2 and B_3 ($= L'_{12}$ or L'_{34}):

$$\begin{aligned} L'^2 = -\Bigg\{ & \frac{1}{\sin^2\theta_2} \frac{\partial}{\partial\theta_2} \left(\sin^2\theta_2 \frac{\partial}{\partial\theta_2} \right) + \frac{1}{\sin^2\theta_2 \sin\theta_1} \frac{\partial}{\partial\theta_1} \left(\sin\theta_1 \frac{\partial}{\partial\theta_1} \right) \\ & + \frac{1}{\sin^2\theta_2 \sin^2\theta_1} \frac{\partial^2}{\partial\phi^2} \Bigg\} \end{aligned} \tag{53}$$

$$L'_{12} = i \frac{\partial}{\partial\theta_0}, \qquad L'_{34} = -i \left\{ -\cos\theta_1 \frac{\partial}{\partial\theta_2} + \cot\theta_2 \sin\theta_1 \frac{\partial}{\partial\theta_1} \right\} \tag{54}$$

4. TRANSFORMATION BETWEEN ENERGIES

Now we derive the relationship between the energies of the hydrogen atom and the 4D harmonic oscillator. We still use dimensionless variables.

The Hamiltonian of the hydrogen atom is $H = -\frac{1}{2}\Delta_x - 1/x$. Using (5), we can reduce this expression to

$$(K_1 + K_3)H = -\frac{1}{2}(K_1 - K_3) - 1 \tag{55}$$

Writing the eigenequation of H

$$H\Psi_n(\mathbf{x}) = e_n \Psi_n(\mathbf{x}) \tag{56}$$

where $\Psi_n(\mathbf{x})(n = 1, 2, \ldots)$ is the eigenfunction, e_n is the corresponding eigenvalue, and

$$e_n = -\frac{1}{2n^2} \qquad \left(\text{in units } \frac{\mu e^4}{\hbar^2} \right) \tag{57}$$

then from equation (55) we obtain

$$\left\{ -\left[\frac{1}{2} + e_n \right] K_1 + \left[\frac{1}{2} - e_n \right] K_3 - 1 \right\} \Psi_n(\mathbf{x}) = 0 \tag{58}$$

Defining the function θ_n by

$$\cosh\theta_n = \frac{1-2e_n}{\sqrt{-8e_n}}, \qquad \sinh\theta_n = -\frac{1+2e_n}{\sqrt{-8e_n}} \tag{59}$$

and using the following relation satisfied by the elements of the $SU(1,1)$ algebra,

$$e^{-iK_2\theta_n}K_3e^{iK_2\theta_n} = K_3\cosh\theta_n + K_1\sinh\theta_n \tag{60}$$

we can rewrite equation (58) as

$$\left\{e^{-iK_2\theta_n}K_3e^{iK_2\theta_n} - \frac{1}{\sqrt{-2e_n}}\right\}\Psi_n(\mathbf{x}) = 0 \tag{61}$$

or

$$\left\{K_3 - \frac{1}{\sqrt{-2e_n}}\right\}e^{iK_2\theta_n}\,\Psi_n(\mathbf{x}) = 0 \tag{62}$$

Equation (62) is an eigenequation of the operator K_3. Thus, we have transformed the eigenequation of H into the eigenequation of the operator K_3.

The Hamiltonian of the 4D harmonic oscillator is $H' = -\frac{1}{2}\Delta_u + \frac{1}{2}u^2$. Its eigenequation can be written as

$$H'\Phi_{n'}(\mathbf{u}) = E_{n'}\Phi_{n'}(\mathbf{u}) \tag{63}$$

where $\Phi_{n'}(\mathbf{u})(n' = 0, 1, \ldots)$ is the eigenfunction, $E_{n'}$ is corresponding eigenvalue, and

$$E_{n'} = (n' + 2)(\text{in units } \hbar\omega) \tag{64}$$

Comparing H' with K_3 expressed in (7), one sees that $K_3 = \frac{1}{2}H'$; thus the eigenequation (63) can be rewritten as

$$K_3\Phi_{n'}(\mathbf{u}) = \frac{E_{n'}}{2}\Phi_{n'}(\mathbf{u}) \tag{65}$$

Equation (65) is also an eigenequation of the operator K_3. Therefore, comparing equation (62) with equation (65), one finds the following mapping:

$$E_{n'} = \sqrt{-\frac{2}{e_n}} \tag{66}$$

$$\Phi_{n'}(\mathbf{u}) = Ne^{iK_2\theta_n}\Psi_n(\mathbf{x}) \tag{67}$$

This establishes the relationship between the energy and energy eigenfunction of the hydrogen atom and those of the 4D harmonic oscillator. N in (67) is a normalization constant, due to the fact that $\Psi_n(\mathbf{x})$ and $\Phi_{n'}(\mathbf{u})$ are normalized in different spaces.

The relation (66) shows that the energy levels of the hydrogen atom and the 4D harmonic oscillator correspond one by one to each other. Substituting equations (57) and (64) into equation (66), we find further

$$n' = 2(n - 1) \tag{68}$$

which shows that the energy levels of the 4D harmonic oscillator connected to the hydrogen atom are always even, and the ground state ($n = 1$) of the hydrogen atom is always mapped into that ($n' = 0$) of the 4D harmonic oscillator.

5. TRANSFORMATION BETWEEN RADICAL FUNCTIONS

Now we express equation (67) using spherical coordinates. In spherical coordinates, the energy eigenfunction $\Psi_n(\mathbf{x})$ of the hydrogen atom can be written as

$$\Psi_n(\mathbf{x}) = R_{nl}(x) Y_{lm}(\theta, \phi) \tag{69}$$

where $R_{nl}(x)$ is the radical function of the hydrogen atom; $n = 1, 2, \ldots$; $l = 0, 1, \ldots, n - 1$, and $m = l, l - 1, \ldots, -l$. The energy eigenfunction $\Phi_{n'}(\mathbf{u})$ of the 4D harmonic oscillator can be written as

$$\Phi_{n'}(\mathbf{u}) = T_{n'l'_2}(u)\, Y_{l'_2, l'_1, m'}(\theta_2, \theta_1, \theta_0) \tag{70}$$

where $T_{n'l'_2}(u)$ is the radical function of the 4D harmonic oscillator, $n' = 0, 1, 2, \ldots$; $l'_2 = n', n' - 2, \ldots, 0$ or 1, depending on whether n' is even or odd; $0 \le l'_1 \le l'_2$, and $m' = l'_1, l'_1 - 1, \ldots, -l'_1$. According to the description in Sections 3 and 4 for the 4D harmonic oscillator connected to the hydrogen atom, one should set $n' = 2(n - 1)$, $l'_2 = 2l$, $l'_1 = l$, and $m' = m$.

Substituting equations (69) and (70) into equation (67) and noting the relation (49) between the three-dimensional and four-dimensional spherical harmonics, one gets

$$T_{n'l'_2}(u) = N_2 e^{iK_2\theta_n} R_{nl}(x) \tag{71}$$

where $N_2 = N/N_1$ is a normalization constant. Equation (71) is just the relationship between the radical functions of the two quantum systems. It shows that using only $\exp(iK_2\theta_n)$, a simple transformation, the radical function of the 4D harmonic oscillator can be obtained from that of the hydrogen atom.

The K_2 in (71) is reduced easily to the form

$$K_2 = i\left(1 + x\frac{d}{dx}\right) = i\left(1 + \frac{1}{2}u\frac{d}{du}\right)$$

Clearly K_2 does not include angular coordinates, and therefore commutes with L_i and L'_α:

$$[K_2, L_i] = [K_2, L'_\alpha] = 0 \tag{72}$$

We noted this fact in deriving relation (71).

The radical function $R_{nl}(x)$ of the hydrogen atom can be expressed as (Qian and Zeng, 1993; Nieto, 1979)

$$R_{nl}(x) = C_{nl}e^{-\rho/2}\rho^l L^{(2l+1)}_{n-l-1}(\rho) \tag{73}$$

$$C_{nl} = 2a_0^{-3/2}n^{-2}[\Gamma(n-l)]^{1/2}[\Gamma(n+l+1)]^{-1/2} \tag{74}$$

where $\rho = (2/n)\,x$, and $L^k_n(\rho)$ is the general Laguerre polynomial. The radical function $T_{n'l'_2}(u)$ of the 4D harmonic oscillator can be expressed as (Kostelecky *et al.*, 1985; Louck and Schaffer, 1960; Louck, 1960a, b)

$$T_{n'l'_2}(u) = C_{n'l'_2}\, e^{-u^2/2}u^{l'_2}\, L^{(l'_2+1)}_{n'/2-l'_2/2}(u^2) \tag{75}$$

$$C_{n'l'_2} = v_0^{-2}\left[2\Gamma\left(\frac{n'}{2} - \frac{l'_2}{2} + 1\right)\right]^{1/2}\left[\Gamma\left(\frac{n'}{2} + \frac{l'_2}{2} + 2\right)\right]^{-1/2} \tag{76}$$

Using equations (73)–(76) and noting $x = \frac{1}{2}u^2$, $n' = 2(n-1)$, and $l'_2 = 2l$, one can check the correctness of equation (71) and determine the normalization constant N_2:

$$N_2 = \left(\frac{a_0}{2}\right)^{3/2} v_0^{-2}n \tag{77}$$

In checking the relation (71), one needs to expand the exponential functions on the right-hand side of (71) as series and use

$$e^{-iK_2\theta_n} = e^{\theta_n}\,e^{1/2\theta_n u d/du}, \qquad e^{\theta_n} = \frac{1}{n} \tag{78}$$

$$L^{(\alpha)}_n(z) = \sum_{k=0}^{n}(-1)^k\binom{n+\alpha}{n-k}\frac{z^k}{k!} \tag{79}$$

In this way, the right side of (71) changes to the same form as the left side of (71). All the calculations in fact are simpler.

Now we give an example. For the ground state ($n = 1$, $l = 0$) of the hydrogen atom, its radical function is

$$R_{10}(x) = 2a_0^{-3/2}e^{-x} \tag{80}$$

Acting with the operator $e^{iK_2\theta_1}$ on $R_{10}(x)$, noting $e^{\theta_1} = 1$, so $\theta_1 = 0$ and $x = \frac{1}{2}u^2$, and multiplying it by the normalization constant $N_2 = (a_0/2)^{3/2}v_0^{-2}$, one obtains the radical function for the ground state of the 4D harmonic oscillator,

$$T_{00}(u) = 2^{1/2}v_0^{-2}e^{-u^2/2} \tag{81}$$

which is the same as the one obtained from equation (75).

6. CONCLUSION

We have discussed the isomorphism between the hydrogen atom and the 4D harmonic oscillator. Setting $\Psi_n(\mathbf{x}) = \Psi_{nlm}(\mathbf{x})$ and $\Phi_{n'}(\mathbf{u}) = \Phi_{n'l'_2l'_1m'}(\mathbf{u})$, we can sum up our results as

$$\Phi_{n'l'_2l'_1m'}(\mathbf{u}) = Ne^{iK_2\theta_n}\Psi_{nlm}(\mathbf{x}) \tag{82}$$

where $N = N_1N_2$, $n' = 2(n - 1)$, $l'_2 = 2l$, $l'_1 = l$, $m' = m$. The results are not only very simple, but also have important applications.

In Zeng *et al.* (1997) we used the isomorphism between a two-dimensional hydrogen atom and a two-dimensional harmonic oscillator to give a q-analogue for the two-dimensional hydrogen atom. In the same way, we can use the isomorphism between a three-dimensional hydrogen atom and a four-dimensional harmonic oscillator to give a q-analogue for the three-dimensional hydrogen atom.

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